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ADAPTIVE CONTROL SYSTEMS WITH
NONLINEAR COMPENSATION

by

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ABSTRACT

This paper considers adaptive control systems with a nonlinear compensator instead of a conventional linear compensator for modification. In order to overcome the intractability of the mathematical equations arising out of the optimization of nonlinear systems subject to random inputs, staircase techniques have been used.

Optimizing equations have been derived for calculating the optimum parameters of the compensator for the least mean-square error between the actual output and the desired output in terms of the statistical properties of the input signal and the plant dynamics. An example of an input-adaptive system has been calculated showing that even a simple no-storage power-series compensator gives a smaller mean-square error than the optimum linear compensator.

1. Introduction

An adaptive or self-optimizing control system may be defined as one which is capable of automatically adjusting its parameters with changes in environment. These changes may be either in the statistical properties of the input function, or in certain parameters of the plant dynamics. The first part of the adaptive process is the identification of these changes. The second part is the modification of certain parameters of a compensator (or controller) which will optimize the performance of the system, based on some specified criteria.

A number of various schemes[12,1*] for the modifying process have been discussed in the past, but in all of these instances a linear controller is used. The controller provides either a variable gain, or some variable time constants that can be used to cancel the undesirable poles of the plant.

The object of this paper is to study the effect of using a nonlinear controller instead of a conventional linear device. Due to the fact that the mathematics of nonlinear systems subject to continuous inputs is rather involved, it will be considered here that the data are sampled regularly and converted into a staircase function, by using a zero-order hold. It may be added that if a digital computer is used for deciding what setting of the controller would give the optimum performance, it is necessary to sample the data. Hence, it is conceivable that the proposed scheme would be suitable for adaptive systems controlled by digital computers.

2. Several Systems for Adaptive Control Using Linear Controllers.

A number of various systems have been proposed for adaptive control

* Numbers indicate references.

with a linear controller. These can be put into two basic types as shown in Figure 1 and 2.

The system shown in Figure 1 is the so-called "model-reference system" developed at M.I.T.[4,6]. This system utilizes a reference model for providing the desired output, which is compared with the actual output. The error is then fed into a computer which determines the modification required in the linear controller in order to optimize the performance. Several configurations have been proposed for the linear controller, but in general, it may be considered as being made up of the parallel combination of a number of lag networks with adjustable gains. In other words, the controller has the following transfer function:

$$G(s) = A_o + \sum_{k=1}^{N} \frac{A_k}{s+a_k} \quad \text{(1)}$$

where the gain constants A_k are controlled by the computer, and the location of the poles at $-a_k$ is fixed on the basis of bandwidth considerations.

The various computations required for the above scheme can be calculated by using staircase functions[7]. These computations can be summarized in the following steps:

(a) Immediate past records of the staircase input $\Gamma x(t)$ and the desired output $\Gamma z(t)$ are sampled $2N + 1$ times. The interval of observation will be taken as $-T_o$ to $T_o + T$, where $T_o = NT$, and T is the interval between each sample. The samples are used to calculate the correlation function ordinates, $\phi_{xx}(kT)$ and $\phi_{xz}(kT)$.

(b) The following coefficients are then calculated:

$$\alpha_k = \sum_{j=0}^{N} \phi_{\Upsilon\Upsilon}(jT) \; \phi_{xx}(\overline{k+j}\; T) \quad \text{(2)}$$

and

$$\beta_k = \sum_{j=0}^{N} \Upsilon(jT) \; \phi_{xz}(\overline{k+j}\; T) \quad \text{(3)}$$

where $\gamma\gamma(kT)$ is the staircase P-response of the plant and $\phi_{\gamma\gamma}(kT)$ are the correlation function ordinates of the plant staircase P-response.

Note: Here it has been assumed that the plant parameters are known. Methods for plant identification will be discussed later.

(c) The computer then solves the following equations for the staircase P-response ordinates u_n of the optimum linear compensator that will give the least mean-square error:

$$\begin{aligned}
&\alpha_o u_o + \alpha_1 u_1 + \alpha_2 u_2 + \ldots\ldots + \alpha_{N-1} u_{N-1} + \alpha_N u_N = \beta_o \\
&\alpha_1 u_o + \alpha_o u_1 + \alpha_1 u_2 + \ldots\ldots + \alpha_{N-2} u_{N-1} + \alpha_{N-1} u_N = \beta_1 \\
&\alpha_2 u_o + \alpha_1 u_1 + \alpha_o u_o + \ldots + \alpha_{N-3} u_{N-1} + \alpha_{N-2} u_N = \beta_2 \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&\alpha_N u_o + \alpha_{N-1} u_1 + \alpha_{N-2} u_2 + \ldots + \alpha_1 u_{N-1} + \alpha_o u_N = \beta_N
\end{aligned} \quad \ldots\ldots(4)$$

(d) From the optimum u's calculated above, the computer solves the following equations to determine the gain-constants A_k:

$$\begin{aligned}
&\Delta_1 A_1 + \Delta_2 A_2 + \Delta_3 A_3 + \ldots\ldots\ldots + \Delta_N A_N = u_1 \\
&\Delta_1 \theta_1 A_1 + \Delta_2 \theta_2 A_2 + \Delta_3 \theta_3 A_3 + \ldots\ldots\ldots + \Delta_N \theta_N A_N = u_2 \\
&\Delta_1 \theta_1^2 A_1 + \Delta_2 \theta_2^2 A_2 + \Delta_3 \theta_3^2 A_3 + \ldots\ldots\ldots + \Delta_N \theta_N^2 A_N = u_3 \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&\Delta_1 \theta_1^{N-1} A_1 + \Delta_2 \theta_2^{N-1} A_2 + \Delta_3 \theta_3^{N-1} A_3 + \ldots\ldots + \Delta_N \theta_N^{N-1} A_N = u_N
\end{aligned} \quad \ldots\ldots(5)$$

$$A_o = u_o \quad \ldots\ldots(6)$$

where

$$\Delta_k = \frac{1-e^{-a_k T}}{a_k} \quad \ldots\ldots(7)$$

and

$$\theta_k = e^{-a_k T} \quad \text{.......................................} (8)$$

a_k being the poles of the compensator, defined in Equation (1).

(e) Finally the computer sends control signals to the controller, adjusting the various gain constants, as required by Equations (5) and (6).

It is presumed that all calculations and adjustments are made in a very short time and that this time is less than the sampling interval; therefore at the next sampling instant the computer is ready for a further modification, depending upon the changes in the statistics of the input as well as the plant dynamics. In the case of slower computers the method would still be valid if during the computation period there has been no change in either the statistics of the input or the plant dynamics.

The linear controller used for this system consists of an attenuator and N lag networks. It may also be noted that 2N + 1 samples of the input and the desired output are required for computing the optimum gain constants. Therefore, a relationship between the number of channels (or lag networks) required in the controller and the extent of correlation between the present value of the signal and its past record is obtained.

The scheme shown in Figure 2 does not require a model and has been used, with slight modifications, in a number of practical systems[12], without computer control. In such cases, the compensator consists of just a variable gain element. Staffin[12] has suggested a controller requiring a variable gain as well as an adjustable compensation network for providing cancellation zeroes to remove the effect of a lightly damped pair of poles.

This method can be used also with a digital controller, but the optimizing equations are more involved than Equation (4). Optimizing equations have been derived in Appendix II.

3. Staircase Techniques for Plant Identification

In the previous section it has been tacitly assumed that the characteristics of the plant are known. In general, this requires some arrangement for the identification of the plant dynamics. A number of different procedures have been proposed using analog computers[8,9]. However, when a digital computer is used for computation, it is more practical to use the same computer for plant identification as well. A staircase technique for plant identification is shown schematically in Figure 3.

This method has the advantage that no test signal is required. It does, however, require measurement of the input to the plant and the output over a number of intervals and computation of the correlation function ordinates $\phi_{rxrx}(kT)$ and $\phi_{rxry}(kT)$. From measurements, the ordinates of the staircase P-response of the Plant $\gamma^{*}(kT)$ can be calculated by solving the following set of simultaneous equations:

$$\sum_{r=o}^{N} \gamma^{*}(rT)\, \phi_{rxrx}(|r-s|T) = \phi_{rxry}(sT) \qquad \text{.................. (9)}$$

$$\text{for } s = 0,\ 1,\ 2, \ldots\ldots N.$$

It may be pointed out that the proposed method does not take into account the error introduced in measurements. If desired, however, these errors could be reduced by making multiple measurements[11] of the plant input and output.

4. Adaptive Systems with Nonlinear Controllers

So far it has been assumed that the controller has a linear structure, capable of adjusting certain gain constants as well as certain time constants. It seems possible that with a nonlinear controller the performance of an adaptive system may be considerably improved. At least one could intuitively argue that since nonlinear filters give smaller mean-square error compared to linear filters subject to random inputs[14], it is expected that nonlinear controllers would give a smaller mean-square error in the case of adaptive systems subject to random variations in the input signal as well as the plant parameters. Moreover, during the interval between the adjustments in the parameters of the controller, a suitably designed nonlinear system would work better than a linear system in many situations.

As a simple example, it will be assumed that the controller is an instantaneous (no-memory) nonlinear device whose output and input are related through the following equation:

$$v(rT) = \sum_{k=o}^{M} a_k f_k \left[x(rT)\right] \qquad (10).$$

For a still simpler case the right-hand side of Equation (10) may be replaced by a power series, e.g.:

$$v(rT) = \sum_{k=1}^{M} a_k x^k(rT) \qquad (11)$$

of which the coefficients a_k are adjustable. The optimum values of a_k which will give the least mean-square error between the desired output and the actual output are obtained by solving the following equations:

$$\begin{aligned}
&\alpha_{11}a_1+\alpha_{12}a_2+ \ldots\ldots\ldots +\alpha_{1M}a_M = \beta_1 \\
&\alpha_{21}a_1+\alpha_{22}a_2+ \ldots\ldots\ldots +\alpha_{2M}a_M = \beta_2 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\alpha_{M1}a_1+\alpha_{M2}a_2+ \ldots\ldots\ldots +\alpha_{MM}a_M = \beta_M
\end{aligned} \qquad (12)$$

where

$$\alpha_{jk} = \sum_{r=o}^{N} \sum_{s=o}^{N} \gamma_r \gamma_s \phi_{\bar{x}^j x^k} (|r-s|T) \qquad (13)$$

and

$$\beta_k = \sum_{r=o}^{N} \gamma_r \phi_{x^k z}(rT) \qquad (14).$$

The nonlinear correlation functions in Equations (13) and (14) are defined as

$$\phi_{x^j x^k}(rT) = \frac{1}{N-r+1} \sum_{p=o}^{N-r} x_p^{\,j} \cdot x_{p+r}^{\,k} \qquad (15)$$

and $$\phi_{x^k z}(rT) = \frac{1}{N-r+1} \sum_{p=o}^{N-r} x_p^{\,k} \cdot z_{p+r} \qquad (16).$$

In general, Equation (10) represents a multipath no-storage nonlinear device, shown schematically in Figure 5.

The optimum values of a_k, which will give the least mean-square error between the actual output and the desired output, are obtained by solving the following sets of equations:

$$\sum_{m=o}^{M} a_m \sum_{r=o}^{N} \sum_{s=o}^{N} \gamma_r \gamma_s \phi_{mj}(f;|r-s|T = \sum_r \gamma_r \quad \phi_{jz}(f;rT) \qquad \text{for } j = o, 1, 2, \ldots\ldots, M \qquad \ldots\ldots(17)$$

where the nonlinear correlation ordinates ϕ_{mj} $(f;|r-s|T)$ and $\phi_{jz}(f;rT)$ are defined as below:

$$\phi_{mj}(f;|r-s|T) = \frac{1}{n-s+1} \sum_{p=o}^{N=s} f_m\left[x_p\right].f_j\left[x_{p+s-r}\right], \quad \text{for } s > r$$

$$= \frac{1}{N-r+1} \sum_{p=o}^{N-r} f_m\left[x_p\right].f_j\left[x_{p+r-s}\right] \quad \text{for } s < r \qquad \ldots\ldots(18)$$

and

$$\phi_{jz}(f;\ rT) = \frac{1}{N-r+1} \sum_{p=o}^{N-r} f_j\left[x_p\right].z_{p+r} \qquad \ldots\ldots(19).$$

In the more general case of a nonlinear device with storage, it is always possible to consider it as a cascade combination of a no-storage nonlinear device and a linear system with storage[15,16]. Hence, the analysis presented in this paper may easily be extended to include such nonlinear devices with storage.

5. Example of an Input-sensing Adaptive Control System:

To compare the relative performances of linear and nonlinear compensators an example of an input-sensing adaptive system is taken. The plant is assumed time-invariant, and the desired output is taken as being equal to the input. As only a small number of samples of the input data are used in a practical case, twenty-one samples of a random signal are assumed as input to the system. The optimum u's for the linear compensator are obtained by solving Equations (4), and these are used to determine the gain-constants for the linear compensator having one attenuator and ten lag networks, by solving Equations (5). The mean-square error for the system with optimum linear compensation is calculated.

A nonlinear compensator is, then, assumed, of the power-series type, with the first five powers. The optimum coefficients for the power series are calculated using Equations (12) and the mean-square error between the actual output and the desired output is evaluated.

Details of the calculations in this example are given in Appendix III.

6. Conclusion:

This example shows that even a simple power-series no-memory nonlinear controller has a performance excelling that of the optimum linear controller. It is, therefore, expected that a nonlinear controller with a finite memory will have a much better performance[14]. A number of such nonlinear controllers will be investigated in a following paper.

It may be added that although a particular form of linear compensation was assumed; this did not enter into the calculation of the optimum u's of the compensator, nor in the mean-square error. Hence, although it is possible to obtain alternative configurations for the linear compensator, the mean-square error cannot be reduced further without using nonlinear devices.

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8. References:

(1) R. F. Kalman, "Design of a Self-Optimizing Control System", Transactions ASME, Vol. 80, p. 468, 1958.

(2) J. A. Aseltine, A. R. Mancini, and C. W. Sarture, "A Survey of Adaptive Control Systems", I.R.E. Transactions on Automatic Control, Vol. AC-6, pp. 102-108, Dec. 1958.

(3) P. R. Stormer, "Adaptive or Self-Optimizing Control Systems - A Bibliography, I.R.E. Transactions on Automatic Control, Vol. AC-6, pp. 65-68, May 1959.

(4) J. J. Bongiorno, "Stability and Convergence Properties of Model-Reference Adaptive Control Systems". I.R.E. Transaction on Automatic Control, Vol. AC-9, pp. 30-41, April 1962.

(5) Edward A. Huber, "A Technique for Adaptive Control of High-Order Systems", I.R.E. Transactions on Automatic Control, Vol. AC-9, pp. 22-29, April 1962

(6) M. L. Moe and G. J. Murphy, "An Approach to Self-Adaptive Control Based on the Use of Time Moments and a Model Reference", I.R.E. Transaction on Automatic Control, pp. 82-92, Oct. 1962.

(7) T. Prasad, "Theory and Application of a Class Nonlinear Staircase Systems", Doctoral Thesis, Cambridge University, England, 1960.

(8) M. Margolis and C. T. Leondes, "A Parameter-tracking Servo for Adaptive Control Systems", I.R.E. Wescon Convention Record, Part IV, 1959.

(9) R. J. McGrath, Rajaraman, and V. C. Rideout, "A Parameter-Perturbation Adaptive Control System", I.R.E. Transactions on Automatic Control, May 1961.

(10) E. G. C. Burt, "Self-Optimizing Servo Systems with Random Inputs", Report presented in Seminar on Nonlinear Control Problems, Cambridge University, England, September 1954.

(11) J. C. Hung, "Theory of Optimum Multiple Measurements", Air Force Research Report No. ARSR 1550, 1961.

(12) Eli Mishkin and L. Braun, "Adaptive Control Systems", McGraw-Hill Book Company, 1961.

(13) S. S. L. Chang, "Synthesis of Optimum Control Systems", Mc-Graw-Hill Book Company, 1961.

(14) N. K. Sinha, "Optimum Nonlinear Filters for Random Signals", I. R. E. International Convention Record, 1962, Circuit Theory.

(15) R. J. Roy and P. M. DeRusso, "A Digital Orthogonal Model for Nonlinear Processes with Two-Level Inputs", I.R.E. Transactions on Automatic Control, pp. 93-101, October 1962.

(16) A. G. Bose, "A Theory of Nonlinear Systems", Research Lab of Electronics, M.I.T., Tech Report 309, 1956.

(17) T. Prasad and V. P. Sinha, "A Digital Procedure for the Study of Nonlinear Systems for Random Processes", to be presented at the Second International Congress of the International Federation for Automatic Control, at Basil, Switzerland, August 1963.

Appendix I: MATHEMATICAL FORMULATION OF STAIRCASE ANALYSIS

1. The theory of staircase analysis has been presented very well by Prasad in his doctoral dissertation (7). However, due to the general non-availability of his work in this country, the theory will be briefly reviewed in this Appendix.

2. The Defining Equations: To define the staircase function, one can use the concept of the well-known zero-order hold circuit, the weighing function for which is given by

$$P(t) = u(t)-u(k-T) \quad \ldots\ldots (A1.1)$$

where $u(t)$ is the unit-step function, and T is the "sampling interval". The function $P(A)$ is shown in Figure A.1.

The extrapolated output of this "clamp" to input $x^*(t)$, shown schematically in Figure A.2, is given by

$$\begin{aligned} \mathcal{S}x(t) &= \int_{-\infty}^{\infty}\left[u(\mathcal{T})-u(\mathcal{T}-T)\right] x^*(t-\mathcal{T})d\mathcal{T} \\ &= \int_{-\infty}^{\infty}\left[u(\mathcal{T})-u(\mathcal{T}-T)\right] \sum_{n=-\infty}^{\infty} x(t-\mathcal{T})\ \delta(t-\mathcal{T}-nT)d\mathcal{T} \\ &= \sum_{n=-\infty}^{\infty} x(nT)\ \left[u(t-nT)-u\ (t-\overline{n+1}T)\right] \end{aligned}$$

or,

$$\mathcal{S}x(t) = \sum_{n=-\infty}^{\infty} x(nT)\ .\ P(t-nT) \quad \ldots\ldots (A1.2)$$

where $P(t-nT)$ represents a rectangular pulse of duration T and unit height, applied at $t-nT$; e.g.,

$$P(t-nT) = \begin{cases} 1, & \text{for } nT \leqslant t \leqslant (n+1)T \\ 0, & \text{elsewhere} \end{cases} \quad \ldots\ldots (A1.3).$$

Equation (A1.2) defines the transformation of the continuous time function $x(t)$ into a staircase function $\mathcal{S}x(t)$, with the following characteristic features:

(a) The value of $\mathcal{S}x$ in a sampling interval remains constant, and equal to the value of $x(t)$ at the commencement of the interval.

(b) If the observation of $x(t)$ is limited to a finite time period, $-T_o$ to T_o+T; where $T_o=NT$, the "staircase function" takes the form

$$\mathcal{S}x(t) = \sum_{m=-N}^{N} x(mT)\ .\ P\ (t-mT) \quad \ldots\ldots (A1.4).$$

(c) It is assumed that the sampling interval T has been selected in compliance with Shannon's Sampling theorems.

Another definition of $\mathcal{J}x(t)$ is given below, which may be given a physical meaning consistent with Figure A1.2.

$$\mathcal{J}x(t) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} x(\sigma).P(t-\sigma)\ \delta(\sigma-nT)d\sigma \quad \text{.............(A1.5)}.$$

3. Properties of the operator P(t):

(i) $$\prod_r P(t-m_r T) = \begin{cases} P(t-mT) \text{ if all } m_r = m \text{ simultaneously} \\ 0, \text{ otherwise} \end{cases} \quad \text{.........................(A1.6)}$$

(ii) $$\int_{-\infty}^{\infty} P(t-iT).P(t-jT).\ \frac{1}{T}\ dt = \Delta_{ij} \quad \text{.....................(A1.7)}$$

where Δ_{ij} = Kronecker delta

(iii) The output of a physically realizable linear system of weighing function w(t), to which the pulse P(t) is applied at t=0, is given by

$$u(t) = \int_{o}^{\infty} w(\tau)\ .P(t-\tau) \quad \text{......................(A1.8)}$$

and $$\mathcal{J}u(t) = \sum_{n=o}^{\infty} u(nT)\ .P(t-nT) \quad \text{......................(A1.9)}.$$

The function $\mathcal{J}u(t)$ will be referred to as the "staircase P-response" of the linear system, and for brevity, u(nT) will be denoted as u_n.

A short table of the staircase P-response of elementary linear systems is given below:

	w(t)	W(s)	u_n $(u_o=o)$
1.	u(t)	$\frac{1}{s}$	T
2.	$\frac{t^{r-1}}{(r-1)!}$	$\frac{1}{s^r}$	$\frac{T^r}{(r)!}\left[n^r-(n-1)^r\right]$
3.	e^{-at}	$\frac{1}{s+a}$	$\frac{1}{a}\ e^{-a(n-1)T}\left[1-e^{-aT}\right]$

4. $\frac{t^{k-1}e^{-aT}}{(k-1)!}$	$\frac{1}{(s+a)k}$	$e^{-a(n-1)T}\sum_{q=o}^{k-1}\frac{T^q(n-1)^q}{a^{k-q}q!} - e^{-anT}\sum_{q=o}^{k-1}\frac{T\ n}{a^{k-q}q!}$
5. $\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$	$\frac{1}{\omega}\left[\cos\omega(n-1)T - \cos\omega nt\right]$
6. $\cos\omega t$	$\frac{s}{s^2+\omega^2}$	$\frac{1}{\omega}\left[\sin\omega nT - \sin\omega(n-1)T\right]$
7. $e^{-at}\sin\omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$	$\frac{1}{r}\left[e^{-a(n-1)T}\sin\left\lvert\omega(n-1)T+\phi\right\rvert\right]$
8. $e^{-at}\cos\ \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$	$\frac{1}{r}\left[e^{-a(n-1)T}\cos\left\lvert\omega(n-1)T+\phi\right\rvert\right]$

where $r=\sqrt{\omega^2+a^2}$, $\phi = \tan\frac{\omega}{a}$.

4. Linear Staircase Systems: It should be noted that

$$x(t-\tau) = \sum_n x(nT).P(t-\tau-nT)$$

and

$$x(t+\tau) = \sum_n x(nT).P(t+\tau-nT) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (A1.10).$$

For the linear system of Figure A1.5,

$$y(t) = \int_o^{\infty} w(\tau)\,\mathcal{S}x(t-\tau)d\tau = \int_o^{\infty} w(\tau)\sum_n x(nT)\ P(t-\tau-nT)d\tau$$

or,
$$y(t) = \sum_n x(nT)\ .\ u(t-nT) = \sum_{n=o}^{\infty} x_n\ u(t-nT) \quad \ldots\ldots\ldots (A1.11)$$

where $u(t-nT) = \int_o^{\infty} w(\tau)\ P\ t-\tau-nT)d\tau$ is the output of the linear system to the input $P(t-nT)$ applied at $t=nT$.

The staircase output is given by

$$\mathcal{S}y(t) = \sum_{k=o}^{\infty} y_k\ P\ (t-kT) = \sum_{k=o}^{\infty}\sum_{r=o}^{\infty} u_r x_{k-r}\ P(t-kT) \quad \ldots\ldots (A1.12)$$

and
$$y_k = u_o x_k + u_1 x_{k-1} + \cdots = \sum_{r=o}^{k} x_r\ u_{k-r} = \sum_{r=o}^{k} x_{k-r}u_r \quad \ldots (A1.13).$$

5. Nonlinear Staircase Systems: Consider the zero-memory (or instantaneous)system of Figure A1.5 having the nonlinear functional $f\left[\ \ \right]$.

Here,
$$\mathcal{S}y(t) = f\left[\mathcal{S}x(t)\right]$$
$$= f\left[\sum_n x_n P(t-nT)\right] = \sum_n f\left[x_n\right] P(t-nT) \quad \ldots\ldots\ldots (A1.14).$$

Similarly, for the system of Figure A1.6

$$\mathcal{J}y(t) = f\left[\sum_n \sum_{r=o}^{N} u_r x_{n-r} P(t-nT)\right] = \sum_n f\left[\sum_{r=o}^{n} u_r x_{n-r}\right] \quad \ldots\text{(A1.15)}$$

Equations (A1.14 and (A1.15) display the following characteristics:

(a) The instantaneous nonlinearity is additive only with respect to the subscript which is symbolic of the sampling instants. For example, if additivity is considered with respect to r in (A1.15), the absurd relation

$$f\left[\sum_n \sum_{r=o}^{n} u_r x_{n-r} P(t-nT)\right] = \sum_{r=o}^{n} f\left[_n\, u_r x_{n-r} P(t-nT)\right]$$

results.

(b) The pulse operator is such that the functional $f\left[\;\right]$ may be considered homogenous, leading to

$$f\left[x_k \cdot P(t-kT)\right] = f\left[x_k\right] \cdot P(t-kT), \quad \ldots\ldots\text{(A1.16)}$$

which is consistent with physical reasoning as well.

Now consider the general no-storage multipath system shown in Figure A1.7.

Here,

$$f\left[\;\right] = \sum_{\mu=o}^{M} a_\mu f_\mu\left[\;\right] \quad \ldots\ldots\text{(A1.17)}$$

$$\therefore \quad \mathcal{J}y(t) = \sum_{\mu=o}^{M} \sum_n a_\mu f_\mu\left[x_n\right] P(t-nT)$$

$$= \sum_{\mu=o}^{M} \sum_n a_\mu,\ f_{\mu,n}\ P(t-nT) \quad \ldots\ldots\text{(A1.18)}$$

where $f_{\mu,n}$ represents the output of the μ^{th} nonlinear path when the input pulse is $x_n = x(nT)$.

If f_μ is replaced by $\left[\;\right]^\mu$, a power-raising device, one obtains the power-series filter, and

$$\mathcal{J}y(t) = \sum_{\mu=1}^{M} a_\mu\left[x_n\right]^m P(t-nT) \quad \ldots\ldots\text{(A1.19)}.$$

Consider the delay-line filter shown in Figure A1.8. The tapping intervals on the delay-line has been taken as T, the same as the sampling interval of the staircase input. The f_μ's are zero-memory nonlinear two-ports, consistent with Equation (A1.14).

The output of the μ^{th} weighing function is given by

$$f_{\mu}\left[x(t-\mu T)\right] = \sum_{n} f_{\mu}\left[x_n\right] P(t-\overline{\mu+n}T)$$

$$= \sum_{n} f_{\mu,n} P(t-\overline{\mu+n}T)$$

$$\mathcal{J}y(t) = \sum_{\mu=o}^{M} \sum_{n} f_{\mu,n} P(t-\overline{\mu+n}T) \quad \text{.....................(A1.20).}$$

We shall now consider a nonlinear system with storage, shown in Figure A1.9, which is made up of an instantaneous nonlinearity $f[\quad]$, followed by a linear system of weighing funtion $w(t)$.

Here,

$$y(t) = \int_{-\infty}^{\infty} w(\tau)\, f\left[\mathcal{J}x(t-\tau)\right] d\tau = \int_{-\infty}^{\infty} w(\tau).\, f\left[\sum_n x_n P(t-nT-\tau)\right] d\tau$$

$$= \sum_{n} f\left[x_n\right].\, u(t-nT) \quad \text{.............................(A1.21)}$$

at $t=kT$, $$y_k = \sum_{r=-\infty}^{k} f\left[x_r\right] u(\overline{k-r}T) \quad \text{.............................(A1.22).}$$

If the linear system is to be physically realizable, (for sampling to commence at $t=o$),

$$\mathcal{J}y(t) = \sum_{k=o}^{\infty} \sum_{r=o}^{k} f\left[x_r\right] u\left(\overline{k-r}\;T\right) .P(t-kT) \quad \text{.............(A1.23)}$$

$$= \sum_{k=o} \sum_{r=o}^{k} u_r .f\left[x_{k-r}\right] P(t\_kT) \quad \text{...............(A1.24).}$$

For the general multipath storage filter shown in Figure A1.10,

$$y(t) = \sum_{i=o}^{M} \int_{-\infty}^{\infty} w_i(\tau)\, f_i\left[\mathcal{J}x\,(t-\tau)\right] d\,\tau$$

$$= \sum_{i=o}^{M} \sum_{n} f_{i,n}\, u_i(t-nT) \quad \text{.........................(A1.25)}$$

and

$$\mathcal{J}y(t) = \sum_{i=o}^{M} \sum_{k} \sum_{r=o}^{k} u_{i,r} f_{i,k-r} P(t-kT) \quad \text{................(A1.26)}$$

where $u_{i,r} = u_i(rT) = 0$ for $r < s$.

6. Nonlinear Staircase Filters of Higher Orders:

Following Zadeh, assume that $f\left[\;\right]$ has a power series expansion of the form

$$f\left[x\right] = \sum_{r=o}^{\infty} a_r x^r \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (A1.27)$$

where it is implied that in the least mean square error sense $f\left[\;\right]$ may be approximated by

$$f\left[x\right] \simeq \sum_{r=o}^{M} a_r x^r \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (A1.28).$$

Hence, the staircase output in the case of the nonlinear filter of "order M" defined by (A1.28), is

$$\mathcal{S}\, y(t) = \sum_{r=o}^{M} \sum_{k=o}^{N} a_r X_k^r \;\; P(t-kT) \ldots\ldots\ldots\ldots\ldots\ldots (A1.29)$$

where $X_k = \sum_{s=o}^{u_k} u_s x_{k-s}$

Equation (A1.29) can be derived alternatively as below:

$$y(t) = a_o + a_1 \int_o^{\infty} w(\tau_1) \sum_{n_1=o}^{N} x_{n1} \; P(t-n_1T-\tau_1) \;\; d\tau_1$$

$$+ a_2 \int_o^{\infty}\int_o^{\infty} w(\tau_1)\, w(\tau_2) \sum_{n_1=o}^{N} P(t-nT-\tau_1)\,.$$

$$\sum_{n_2=o}^{N} x_{n2} \; P(t-n_2T-\;\;_2) d\tau_1 d\tau_2$$

$$+ \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.$$

$$+ a_M \int_o^{\infty} \cdots \int_o^{\infty} w(\tau_1) \ldots w(\tau_M) \sum_{n_1=o}^{N} X_{n1} P(t-n_1T-\tau_1) \ldots\ldots$$

$$\sum_{n_M=o}^{N} x_{n_M} \,.\, P(t-n_MT-\tau_M) \;\; d\tau_1 \ldots\ldots d\tau_M$$

$$= a_o + a_1 \sum_{n_1=o}^{N} x_{n1} \; u(t_1-n_1T)$$

$$+ a_2 \sum_{n_1=o}^{N} x_{n_1} u(t-n_1T) \cdot \sum_{n_2=o}^{N} x_{n_2} u(t-n_2T)$$

$$+ \ldots\ldots\ldots\ldots\ldots$$

$$+a_M \sum_{n_1=o}^{N} x_{n_1} u(t-n_1T) \ldots \sum_{n_M=o}^{N} x_{n_M} u(t-n_MT) \quad \ldots\text{(A1.30)}.$$

Note that at t=kT, the coefficient of a_r is

$$\sum_{n_1=o}^{N} x_{n1} u(kT-n_1T) \quad \ldots\ldots \quad \sum_{n_r=o}^{N} x_{nr} u(kT-n_rT) \quad = \quad X_k^r$$

which leads to

$$\text{Ш}\, y(t) = \sum_{r=o}^{M} \sum_{k=o}^{N} a_r X_k^r P(t-kT) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(A1.31)}.$$

Appendix II: OPTIMIZATION OF LINEAR AND NONLINEAR STAIRCASE SYSTEMS

1. The theory discussed in Appendix I will be used to derive the various optimizing equations used in this paper.

2. Model Reference System with a Linear Controller: Consider the model reference system shown in Figure 1.

The linear controller has its staircase P-response given by $\mathcal{J}u(t)$, and the plant is described by its staircase P-response $\mathcal{J}\,\Upsilon(t)$.

Using equation (A1.13), the staircase output of the controller is given by

$$\mathcal{J}v(t) = \sum_{k=o}^{\infty} \sum_{r=o}^{k} u_r\, x_{k-r}\, P(t-kT) \quad \text{.......................(A2.1)}$$

and

$$v_k = \sum_{r=o}^{k} u_r\, x_{k-r}\,. \quad \text{..................................(A2.2)}$$

Similarly, the staircase output of the Plant is given by

$$\mathcal{J}\,y(t) = \sum_{n=o}^{\infty} \sum_{r=o}^{k} \Upsilon_r\, v_{n-r}\, P(t-nT)$$

or,

$$\mathcal{J}\,y(t) = \sum_{n=o}^{\infty} \sum_{k=o}^{N} \sum_{r=o}^{k} u_r x_{k-r}\, \Upsilon_{n-k} P(t-nT) \quad \text{................(A2.3)}$$

and

$$y_n = \sum_{k=o}^{N} \sum_{r=o}^{k} u_r x_{k-r}\, \Upsilon_{n-k}\,. \quad \text{......................(A2.4)}$$

The mean-square error between the desired output $\mathcal{J}z(t)$ and the actual output $\mathcal{J}\,y(t)$ is given by

$$\overline{\epsilon^2} = \frac{1}{N} \sum_{n=o}^{N} \left[z_n - y_n\right]^2 \quad \text{............................(A2.5)}$$

where N = total number of samples.

With considerable algebraic manipulation, equation (A2.5) can be transformed as

$$\overline{\epsilon^2} = \phi_{\mathcal{J}z\,\mathcal{J}z}(o) - 2 \sum_{n=o}^{N} u_n \beta_n + \sum_{r=o}^{N} \sum_{s=o}^{N} u_r u_s\, \alpha_{|r-s|} \quad \text{....(A2.6)}$$

where

$$\phi_{\mathcal{J}z\,\mathcal{J}z}(o) = \frac{1}{N} \sum_{n=o}^{N} z_n^2 \quad \text{.....................(A2.7)}$$

$$\alpha_k = \sum_{j=o}^{N} \phi_{\Gamma\gamma\Gamma\gamma}(jT)\ \phi_{\Gamma x \Gamma x}(\overline{k+j}T) \qquad (A2.8)$$

$$\beta_k = \sum_{j=o}^{N} \gamma(jT)\ \phi_{\Gamma x \Gamma z}(\overline{k+j}T) \qquad (A2.9)$$

and

$$\phi_{\Gamma\gamma\Gamma\gamma}(jT) = \sum_{k=o}^{N-j} \gamma_k \gamma_{k+j} \qquad (A2.10).$$

The optimum u's, for minimum mean-square error are obtained by setting the various partial derivates $\frac{\partial \overline{\epsilon^2}}{\partial u_r}$ if Equation (A2.6) to zero. These give the set of simultaneous equations for the optimum u's as in Equation (4).

Once the optimum u's are obtained, substituting these back into Equation (A2.6) gives the minimum mean-square error for the linear case,

$$\overline{\epsilon^2}_{min} = \phi_{\Gamma z \Gamma z}(o) - \sum_{n=o}^{N} u_n \beta_n \qquad (A2.11)$$

The next problem is to determine the linear network the staircase P-response for which would correspond to the optimum u's obtained. Since only a finite number of these u's are known, a number of different configurations are possible. For instance, one may assume that the transfer function of the linear controller is given by

$$G(s) = A_o + \sum_{k=1}^{N} \frac{A_k}{s+a_k} \qquad (A2.12)$$

where A_o represents a variable attenuator, and A_k represents variable gain constants. The location of the poles, A_k, is arbitrary, except that they should not be equal. However these may normally be fixed on considerations of bandwidth.

Using the table in Appendix I, the staircase P-response for the transfer function given by Equation (A2.12) is obtained as

$$u_o = A_o$$

$$u_r = \sum_{k=1}^{N} A_k \frac{1-e^{-a_kT}}{a_k} \cdot e^{-(r-1)a_kT} \qquad (A2.13).$$

Equations (A2.13) may be arranged as in Equations (5) and (6) for solving A_k.

3. Linear Controller for System without a Reference Model:

For the system shown in Figure 2,

$$y_n = \sum_{k=o}^{N} \sum_{r=o}^{k} u_r \epsilon_{k-r} \gamma_{n-k} \quad \text{.......................(A2.14)}$$

and

$$z_n = \sum_{j=o}^{N} \sum_{k=o}^{j} \sum_{r=o}^{k} u_r \epsilon_{k-r} \gamma_{j-k} \; h_{n-j} \quad \text{...................(A2.15).}$$

Also, $\epsilon_n = z_n - x_n$ ....(A2.16).

Equations (A2.15) and (A2.16) may be used to eliminate ϵ_n, giving and explicit relationship between z_n and x_n . Finally one may find the optimum values of u_r by taking partial derivatives of the expression for the mean-square error. However, the algebra gets very involved, and so, this will not be pursued further in this paper.

4. Optimizing Equations for a System with Nonlinear Compensation:

Consider the system shown in Figure 4, the nonlinear controller being defined by Equation (10). The staircase output of the controller is obtained by noting that

$$v_n = \sum_{k=o}^{M} a_k f_k \left[x_n\right] \quad \text{....................................(A2.17).}$$

The staircase output of the system is, therefore, given by

$$\mathcal{J}y(t) = \sum_{k=o} \sum_{r=o} \gamma_r \, v_{k-r} \quad P(t-kT) \quad \text{......................(A2.18)}$$

Using the expression for v_n from Equation (A2.17) the output at the n^{th} sampling instant is given by

$$y_n = \sum_{r=o}^{M} \sum_{k=o}^{n} a_r f_r \left[x_n\right] \gamma_{n-k} \quad \text{.........................(A2.19).}$$

The mean-square error is given by

$$\overline{\epsilon^2} = \frac{1}{N} \sum_{n=o}^{N} \left(z_n - y_n\right)^2$$

$$= \phi_{\mathcal{J}z\mathcal{J}z}(o) - \frac{1}{N} \sum_{n=o}^{N} 2z_n y_n + \frac{1}{N} \sum_{n=o}^{N} y_n^2 \quad \text{...........(A2.20).}$$

With considerable algebraic manipulation, equation (A2.20) can be put into the following form:

$$\overline{\epsilon^2} = \phi_{\Gamma z \Gamma z}(o) - 2 \sum_{m=o}^{M} a_m \sum_r \Upsilon_r \, \phi_{mz} (f;rT)$$

$$+ \sum_m \sum_n a_m a_n \sum_r \sum_s \Upsilon_r \Upsilon_s \, \phi_{mn}(f; |r-s| T) \quad \text{.........(A2.21)}$$

where the nonlinear correlation ordinates are defined as below:

$$\phi_{mz}(f;rT) = \frac{1}{N-r+1} \sum_{p=o}^{N-r} f_m \left[x_p\right] \cdot z_{p+r} \quad \text{............(A2.22)}$$

and

$$\phi_{mn}(f; |r-s|T) = \frac{1}{N-s+1} \sum_{p=o}^{N-s} f_m \left[x_p\right] \cdot f_n \left[x_{p+s-r}\right], \quad \text{for } r < s$$

$$= \frac{1}{N-r+1} \sum_{p=o}^{N-r} f_m \left[x_p\right] \cdot f_n \left[x_{p+r-s}\right] \quad \text{for } r > s \quad \text{............(A2.23).}$$

Finally, the optimum values of a_r may be obtained by setting the partial derivatives $\dfrac{\partial \overline{\epsilon^2}}{\partial a_r} = 0.$ The optimizing Equations (17) result.

The minimum value of the mean-square error is given by

$$\overline{\epsilon^2}_{min} = \phi_{\Gamma z \Gamma z}(o) - \sum_{m=o}^{M} a_m \sum_r \Upsilon_r \, \phi_{mz} (f;rT) \quad \text{.......(A2.24)}$$

Appendix III: EXAMPLE OF AN ADAPTIVE SYSTEM WITH NONLINEAR COMPENSATION

As an example, a time-invariant plant is assumed with transfer-function given by $\frac{1}{s+1}$. For a sampling interval T=0.1 second, the staircase P-response of plant is given by

$$\Upsilon(0) = 0$$

$$\Upsilon(nT) = e^{-(n-1)T} . (1-e^{-T}) = e^{-0.1(n-1)}(1-e^{-0.1}) .$$

The first eleven of the staircase P-response ordinates are given below:

$$\Upsilon(0) = 0$$
$$\Upsilon(T) = 0.095163$$
$$\Upsilon(2T) = 0.086106$$
$$\Upsilon(3T) = 0.077913$$
$$\Upsilon(4T) = 0.070498$$
$$\Upsilon(5T) = 0.063789$$
$$\Upsilon(6T) = 0.057719$$
$$\Upsilon(7T) = 0.052227$$
$$\Upsilon(8T) = 0.047256$$
$$\Upsilon(9T) = 0.042759$$
$$\Upsilon(10T) = 0.038691$$

The random signal input to this plant is given below:

t	x(t)
-10T	1.05
-9T	2.24
-8T	2.41
-7T	4.22
-6T	3.76
-5T	7.79
-4T	9.96
-3T	9.63
-2T	8.96
-T	8.54
0	5.86
T	2.89
2T	6.36
3T	0.94
4T	1.04
5T	0.71
6T	5.11

7T	0.24
8T	0.10
9T	5.22
10T	0.71 .

The desired output is taken as equal to the input, i.e., z(t)=x(t).

To determine the optimum linear compensator, the various correlation functions are calculated and used to calculate the various α's and β's defined in Equations (2) and (3). These are given below:

$\alpha_0 = 5.150164$ $\beta_0 = 12.84787$
$\alpha_1 = 4.785945$ $\beta_1 = 12.03603$
$\alpha_2 = 4.495699$ $\beta_2 = 11.10204$
$\alpha_3 = 4.158237$ $\beta_3 = 10.08288$
$\alpha_4 = 3.752778$ $\beta_4 = 9.11111$
$\alpha_5 = 3.375715$ $\beta_5 = 8.32439$
$\alpha_6 = 3.106233$ $\beta_6 = 7.62324$
$\alpha_7 = 2.861431$ $\beta_7 = 6.77098$
$\alpha_8 = 2.561063$ $\beta_8 = 6.27139$
$\alpha_9 = 2.399077$ $\beta_9 = 5.67021$
$\alpha_{10} = 2.190972$ $\beta_{10} = 4.91666$.

These are used in Equations (4) to calculate the staircase P-response ordinates of the optimum linear controller. These are -

$u_0 = 2.418129$
$u_1 = 0.4830323$
$u_2 = 0.1854337$
$u_3 = 0.3234086$
$u_4 = 0.1357262$
$u_5 = 0.2605517$
$u_6 = 0.3003115$
$u_7 = -0.5957485$
$u_8 = 0.4392162$
$u_9 = 0.1573702$
$u_{10} = -0.3836328$.

The mean-square error between the actual output and the desired staircase output for the optimum linear filter, calculated by using Equation (A2.11) is found to be 4.45086.

Next, a power-series controller is assumed, the input and output for which are related through Equation (11), with M = 5.

The nonlinear correlation ordinates are computed as in Equations (15) and (16) and the coefficients a_k obtained as below:

$$a_1 = 9.270819$$
$$a_2 = -2.219032$$
$$a_3 = 0.009191157$$
$$a_4 = 0.0155692$$
$$a_5 = -0.000012995 \ .$$

With these coefficients, the mean-square error is found to be 3.0306, compared with 4.45086 for the linear case.

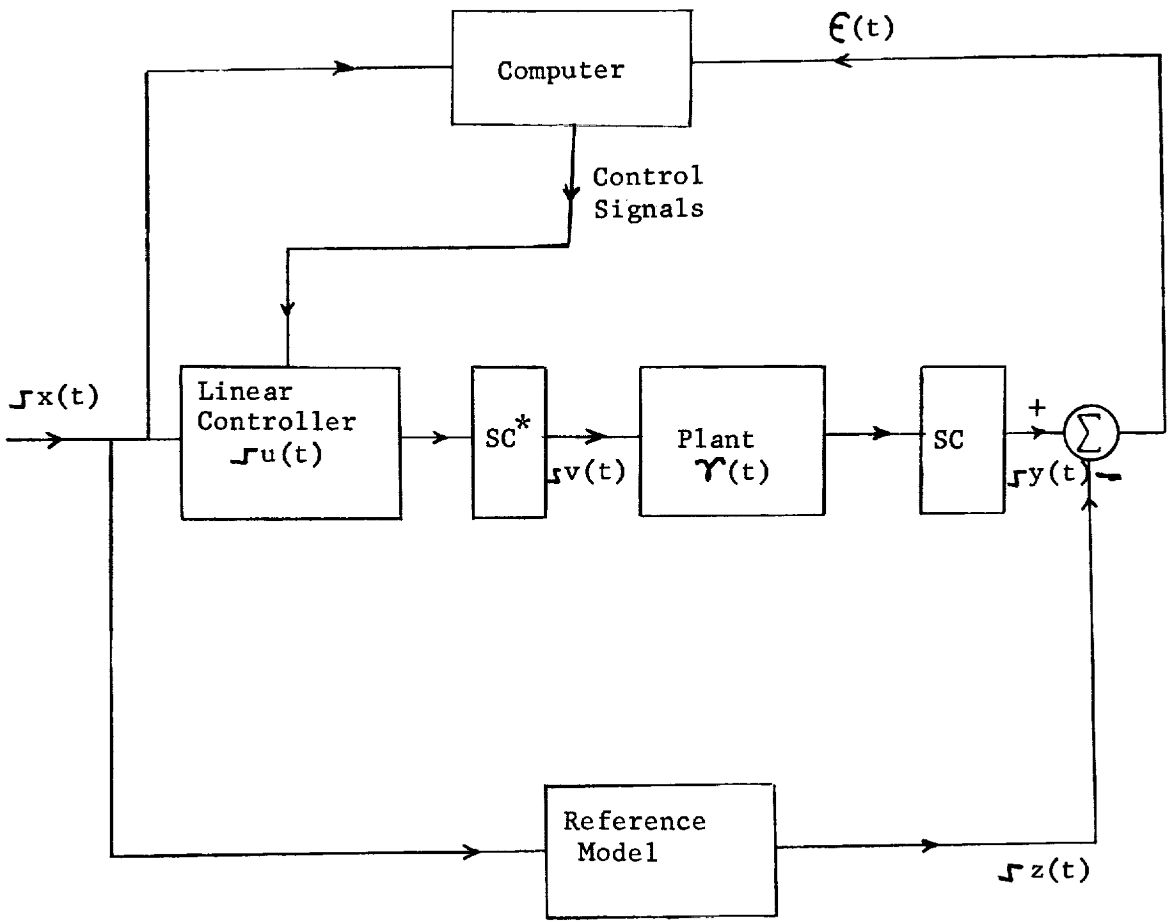

Figure 1: The Model Reference Type Adaptive System.

*SC = Sampler and Clamp

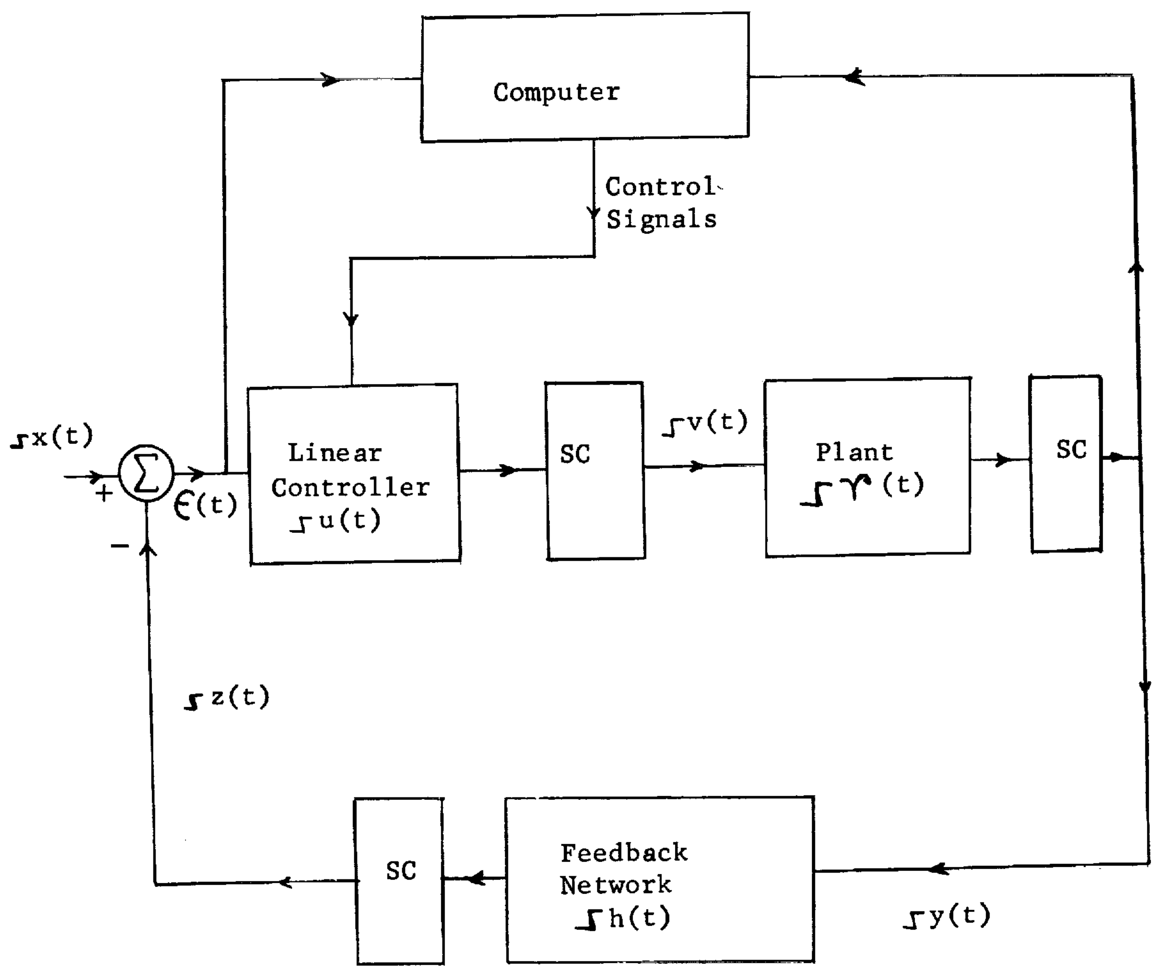

Figure 2: Adaptive System not Requiring a Model

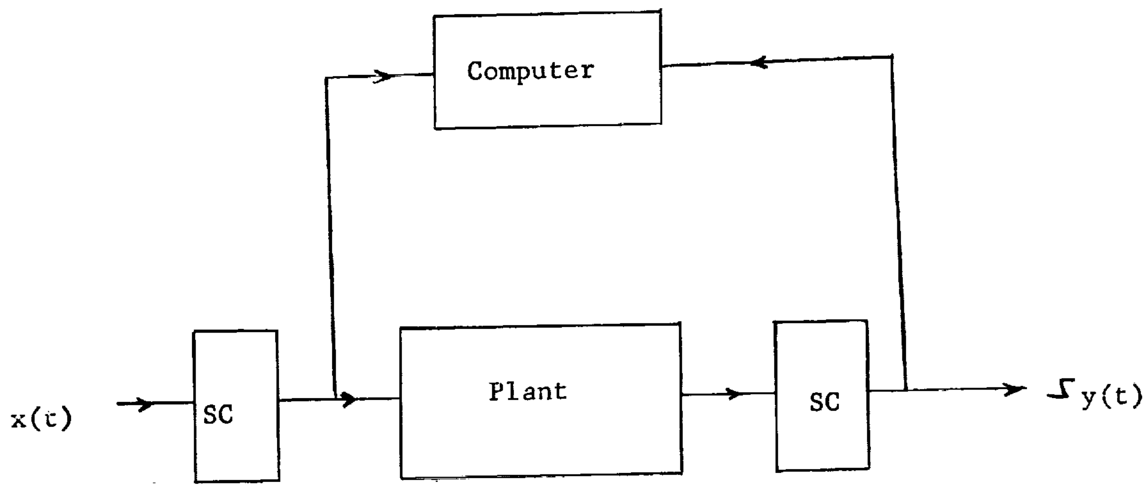

Figure 3: Scheme for Plant Identification.

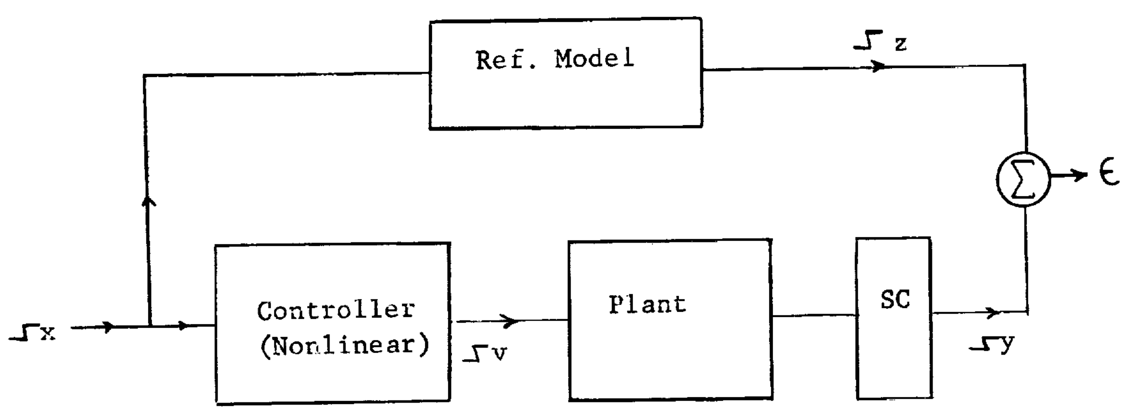

Figure 4: Nonlinear Controller without storage.

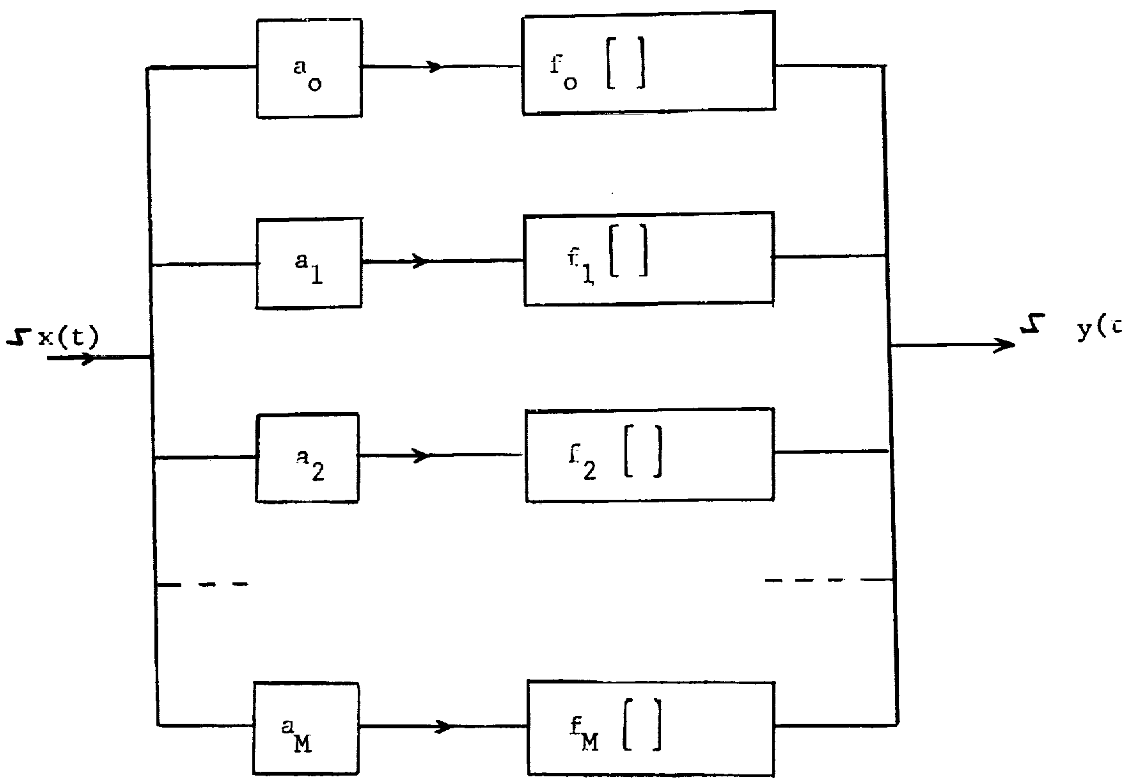

Figure 5: A Multipath No-storage Nonlinear Controller

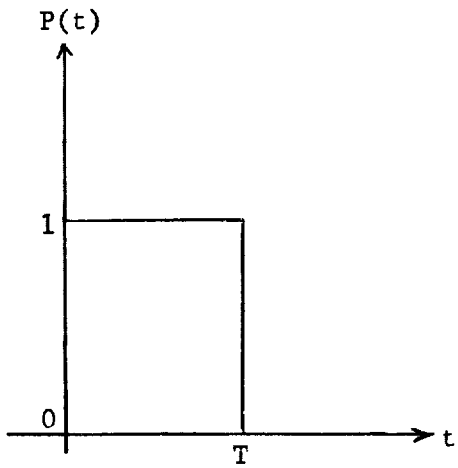

Figure A1.1

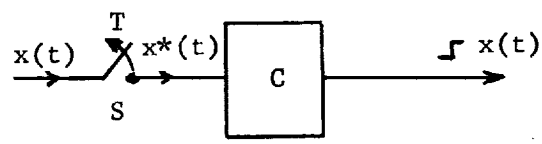

Figure A1.2

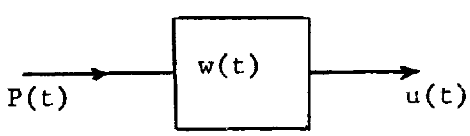

Figure A1.3

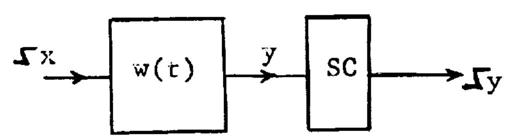

Figure A1.4

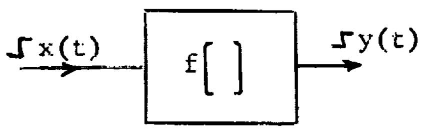

Figure A1.5

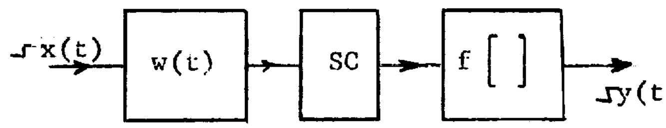

Figure A1.6

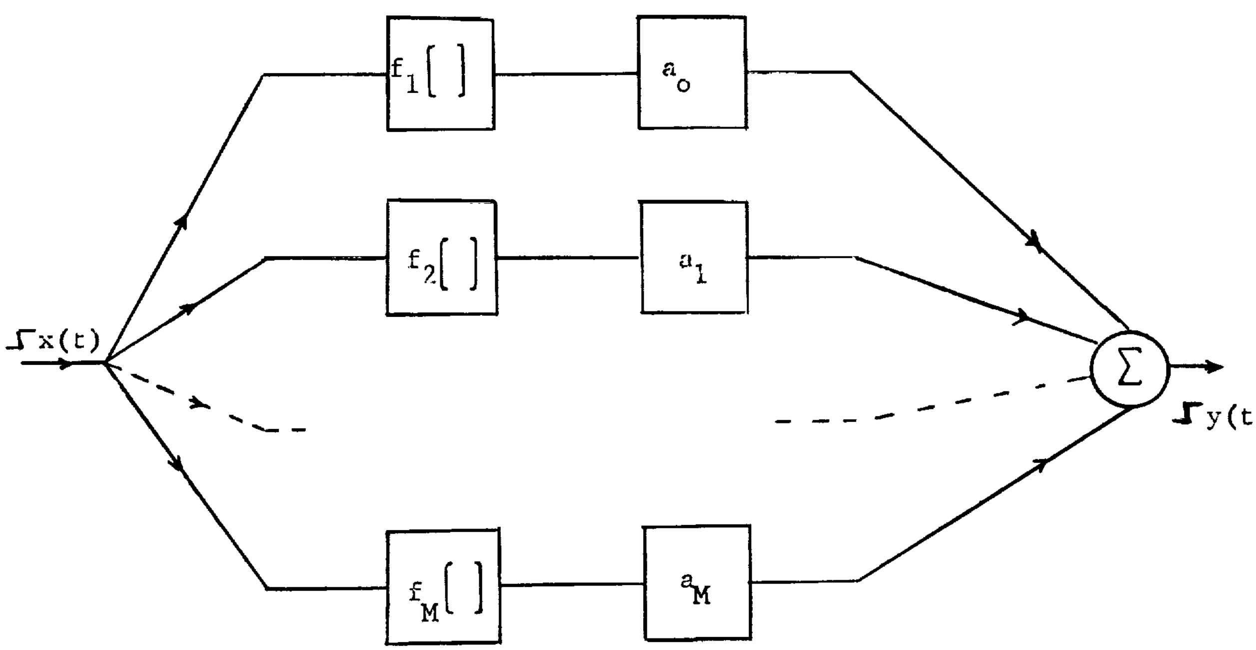

Figure A1.7

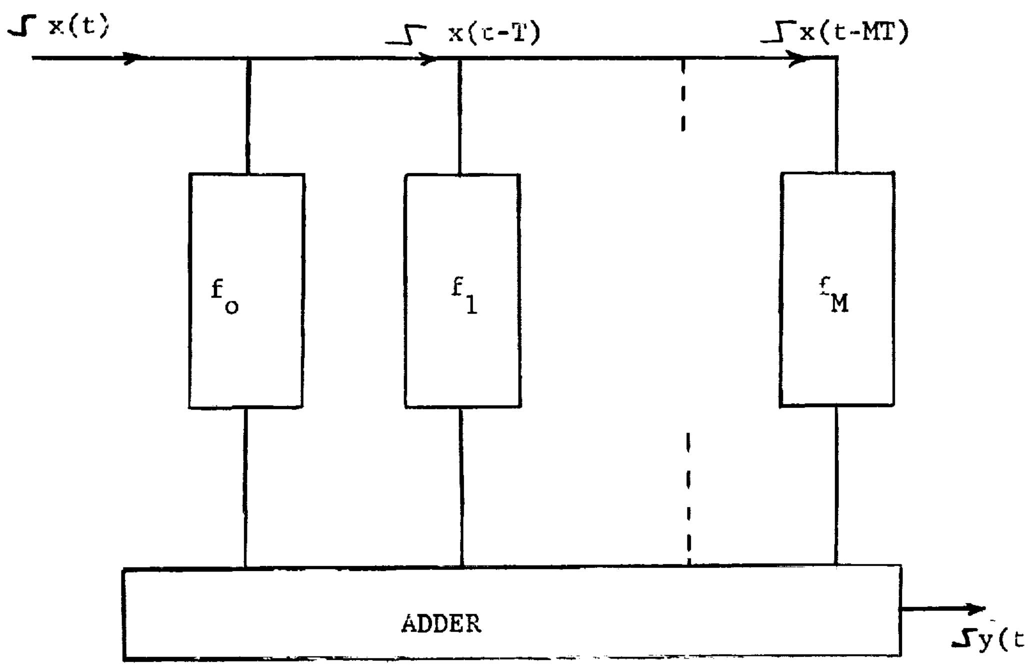

Figure A1.8

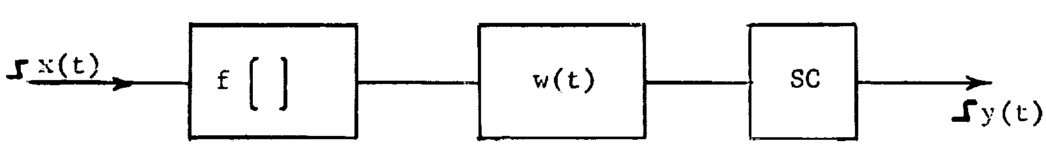

Figure A1.9

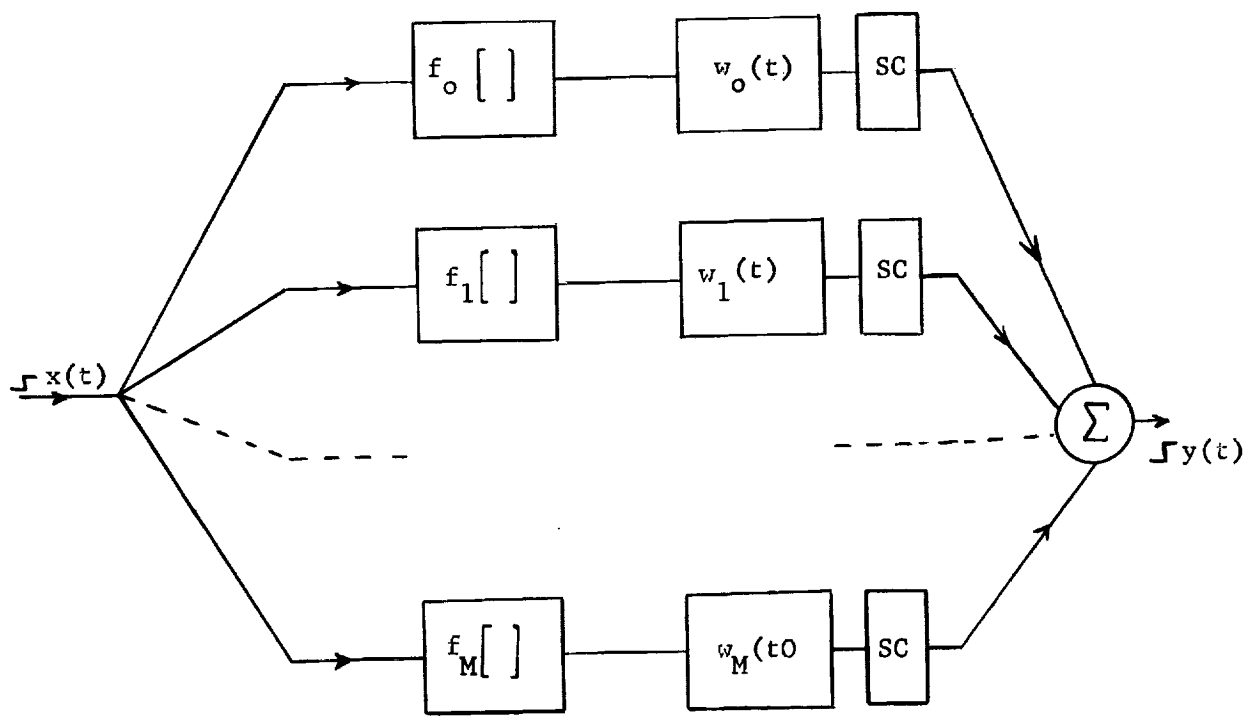

Figure A1.10

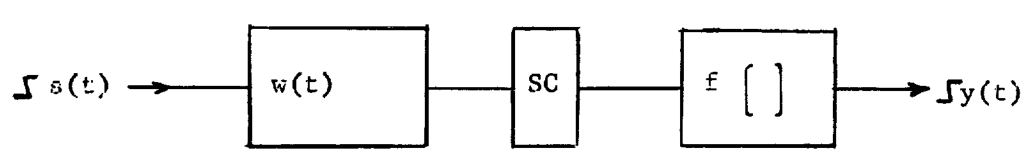

Figure A1.11